

Evolution equation of entanglement for multi-qubit systems

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We discuss entanglement evolution of a multi-qubit system when one of its qubits is subjected to a general noisy channel. For such a system, an evolution equation of entanglement for a lower bound for multi-qubit concurrence is derived. Using this evolution equation, the entanglement dynamics of an initially mixed three-qubit state composed of a GHZ and a W state is analyzed if one of the qubits is affected by a phase, an amplitude or a generalized amplitude damping channel.

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I. INTRODUCTION

Entanglement is an exclusive nonclassical resource of quantum mechanics with promising applications in communication, cryptography and computing [1]. In many of these applications, this resource is often required to be distributed between different partners; a request that implies that at least one physical (sub-)system is to be transmitted through a communication channel. In general, such a coupling of a quantum system to some environmental channel leads to decoherence of the system and usually to some loss of entanglement before the state of the system can be further utilized. Therefore, it is of great practical importance to investigate the decoherence and time evolution of entanglement for quantum systems that undergo the action of noisy channels.

Typically, the time evolution of entanglement of a system is deduced from studying its state evolution under the influence of decoherence [2–5]. Following this line, we have recently analyzed for example the entanglement dynamics of initially prepared pure three-qubit GHZ and W states if transmitted through one of the Pauli channels σ_z , σ_x , σ_y or a depolarizing channel [5]. In this work, we employed the analytical solutions of the master equation as obtained by Jung *et al.* [6] and a lower bound for the three-qubit concurrence [7] in order to quantify the time-dependent entanglement of the mixed three-qubit states.

Instead of making explicit use of the state evolution for the analysis of the entanglement dynamics of a given system, Konrad *et al.* [8] recently derived an evolution equation of entanglement for a *two-qubit* system that provides a direct relationship between the initial and the final entanglement of the system when one of its qubits is subjected to an arbitrary noise. Subsequently, Li *et al.* [9] derived a generalized evolution equation of en-

tanglement for a (finite dimensional) *bipartite* system, if initially prepared in a pure state and affected by an arbitrary noisy channel. Recently, moreover, this latter result [9] has been extended to the case of an initial mixed state of a bipartite system [10, 11].

In this work, we suggest an evolution equation of entanglement for a *multi-qubit* system when one of its qubits undergoes the action of an arbitrary channel, which is given by a completely positive (non-)trace-preserving map. Since there is no an analytically computable measure of entanglement for mixed multi-qubit states [12], we shall make use of the lower bound by Li *et al.* [7] for the multi-qubit concurrence. In fact, this lower bound was proven to satisfy the general requirements [3] of an entanglement measure [7, 13]. Therefore, the evolution equation derived below is capable to characterize the entanglement dynamics of multi-qubit states representing a lower bound for an actual currently unavailable entanglement evolution. As example, we shall discuss in details the time evolution of entanglement of an initially mixed three-qubit state composed of a GHZ and a W state [14]

$$\rho(p) = p |GHZ\rangle \langle GHZ| + (1-p) |W\rangle \langle W|, \quad (1)$$

if one of the qubits is affected by a phase or an amplitude damping channel. For a generalized amplitude damping channel, moreover, we also show the sudden death of entanglement [4] for this mixed three-qubit state.

The paper is organized as follows. In the next section, we first recall how the entanglement of a multi-qubit system can be quantified either by means of the N -qubit concurrence [3, 12] or in terms of a lower bound to this measure as suggested by Li *et al.* [7]. This lower bound was constructed in such a way that it includes only bipartite concurrences according to some “bi-partite” cuts of the multi-qubit system. In Section. II B, we display and discuss the evolution equation of bipartite concurrence as obtained by Li *et al.* [9]. Based on their results for the bipartite concurrence, we then derive an evolution equation of the lower bound to the concurrence for a three-qubit

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system in Section II.C, if one of its qubits is affected by an arbitrary noisy channel and if we start with an initially pure state. In Section II.D, we shall discuss also possible extensions of this equation to the cases of N qubits, initially mixed states of three qubits, and if a system is affected by many-sided noisy channels, i.e. when several of its qubits undergo simultaneously the action of some local noise. In Section III, we later analyze in detail the entanglement dynamics of the initially three-qubit mixed state (1) for three realistic noise models, namely, a phase damping, an amplitude damping and a generalized amplitude damping noise. Finally, a conclusion is drawn in Section IV.

II. EVOLUTION EQUATION OF THE LOWER BOUND FOR MULTI-QUBIT CONCURRENCE

Until the present, it has been found difficult to quantify the entanglement of mixed many-partite states, and no general solution is known [12] apart from Wootters' (two-qubit) *concurrence* [15]. This concurrence provides indeed a very powerful measure of entanglement but is just suitable for two-qubit systems. Various extensions of Wootters' concurrence have been worked out over the years, and especially for mixed bipartite states, if the dimensions of the associated Hilbert (sub-)spaces are larger than two [12, 16, 17]. Nonetheless, a full generalization of the concurrence towards mixed many-partite states has remained a challenge until now and will require further studies [12]. In practice, however, one is mostly interested in the minimum (nontrivial) amount of entanglement [18] which is preserved in a mixed state of some system when coupled to its environment, for instance, due to transmission of one or several of its subsystems through a communication channel. To this end, various analytically computable lower bounds to concurrence were suggested recently [3, 7, 19]. In the next section, we shall exploit the lower bound to the multi-qubit concurrence as suggested by Li *et al.* [7]. As mentioned before, this bound is based on the bipartite concurrences of the multi-qubit system with regard to some bipartite cuts. This "bi-partitioning" enables us eventually to construct an evolution equation for multi-qubit systems which is based on the evolution equation for bipartite systems [9].

A. A lower bound for N -qubit concurrence

For a given *pure* N -qubit state $|\psi\rangle$, the concurrence can be written as [7, 20]

$$C_N(|\psi\rangle) = \sqrt{1 - \frac{1}{N} \sum_{i=1}^N \text{Tr} \rho_i^2}, \quad (2)$$

and where the $\rho_i = \text{Tr} |\psi\rangle \langle \psi|$ denote the reduced density matrix of the i -th qubit which is obtained by tracing out

the remaining $N - 1$ qubits. Since any *mixed* state can be expressed also as a convex sum of some pure states $\{|\psi_i\rangle\}$: $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, the definition (2) for the concurrence for pure N -qubit states can be generalized for mixed states by means of the so-called *convex roof* [12]

$$C_N(\rho) = \min \sum_i p_i C_N(|\psi_i\rangle). \quad (3)$$

In this latter expression, however, the minimum has to be found with regard to *all possible* decompositions of ρ into pure states $|\psi_i\rangle$.

Unfortunately, no solution has been found so far to optimize the concurrence (3) of a multi-qubit system analytically [12], apart from the two-qubit systems [15] and a special case of three qubits [14]. Instead of the computationally demanding optimization of the right-hand side of Eq. (3), a much simpler lower bound to this measure has been suggested recently by Li *et al.* [7]

$$C_N(\rho) \geq \tau_N(\rho) \equiv \sqrt{\frac{1}{N} \sum_{n=1}^N \sum_{k=1}^K (C_k^n)^2}. \quad (4)$$

This bound is defined in terms of the N "bi-partite" concurrences C^n [13] that correspond to the possible (bipartite) cuts of the multi-qubit system in which just one of the qubits is discriminated from the other $N - 1$ qubits. For the separation of the n -th qubit, the bipartite concurrence C^n is given by a sum of $K = 2^{N-2}(2^{N-1} - 1)$ terms C_k which are expressed as

$$C_k^n = \max\{0, \lambda_k^1 - \lambda_k^2 - \lambda_k^3 - \lambda_k^4\}, \quad (5)$$

and where the λ_k^m , $m = 1..4$ are the square roots of the four nonvanishing eigenvalues of the matrix $\rho \tilde{\rho}_k^n$, if taken in decreasing order. These (non-hermitian) matrices $\rho \tilde{\rho}_k^n$ are formed by means of the density matrix ρ and its complex conjugate ρ^* , and are further transformed by the operators $\{S_k^n = L_k^n \otimes L_0, k = 1, \dots, K\}$ as: $\tilde{\rho}_k^n = S_k^n \rho^* S_k^n$. In this notation, moreover, L_0 is the (single) generator of the group $\text{SO}(2)$, while the $\{L_k^n\}$ are the $K = 2^{N-2}(2^{N-1} - 1)$ generators of the group $\text{SO}(2^{N-1})$. We note that the lower bound (4) reduces to the Wootters concurrence [15] for just two qubits. For details about the explicit construction of the lower bound (4) we refer to [7].

Let us display this lower bound (4) especially for three-qubits, $\tau_3(\rho)$, for which we wish later to describe the entanglement dynamics. For such states, the lower bound $\tau_3(\rho)$ can be written in terms of the three bipartite concurrences that correspond to possible cuts of the two qubits from the remaining one, i.e.

$$\tau_3(\rho) = \sqrt{\frac{1}{3} \sum_{k=1}^6 (C_k^{12|3})^2 + (C_k^{13|2})^2 + (C_k^{23|1})^2}. \quad (6)$$

The bipartite concurrence $C_k^{ab|c}$ (for $a, b, c = 1..3$ and $a \neq b \neq c \neq a$) are obtained as described above with the

help of the operators $\{S_k^{ab|c} = L_k^{ab} \otimes L_0^c, k = 1 \dots 6\}$, where L_0 is the generator of the group $SO(2)$ which is given by the second Pauli matrix $\sigma_y = -i(|0\rangle\langle 1| + |1\rangle\langle 0|)$. The (six) generators L_k^{ab} of the group $SO(4)$ that can be expressed explicitly by means of the totally antisymmetric Levi-Cevita symbol in four dimensions as $(L_{kl})_{mn} = -i\varepsilon_{klmn}$; $k, l, m, n = 1 \dots 4$ [21].

B. Evolution equation for bipartite systems

Having the explicit expression (4) for the lower bound $\tau_N(\rho)$ to the multi-qubit concurrence, we can derive an evolution equation of this entanglement measure. Since the lower bound $\tau_N(\rho)$ is given just in terms of the bipartite concurrences C^n , let us first reconsider the evolution equation for a bipartite system as suggested by Li *et al.* [9]. Suppose, $|\chi\rangle$ is a pure state of a bipartite system $d_1 \otimes d_2$ with dimension d_1 and d_2 of the corresponding subsystems, and the second subsystem undergoes the action of a noisy channel \mathcal{S} . Then, the final state of the system is a mixed state in general and takes the form $\rho = (1 \otimes \mathcal{S})|\chi\rangle\langle\chi|$. On the other hand, any pure state $|\chi\rangle$ can be obtained also from the maximally entangled state $|\phi\rangle = \sum_{i=1}^{d_2} |i\rangle \otimes |i\rangle / \sqrt{d_2}$ of the bipartite system by $|\chi\rangle = (M \otimes 1)|\phi\rangle$ [9]. In this notation, M denotes a local (filtering) operator [22, 23] that acts on the first subsystem of the maximally entangled state. Although local operations, usually associated with projective measurements or a passage through a channel, cannot change entanglement because of its nonlocal nature, there is a special case of a stochastic local operation, a filtering operation, that can be employed to influence on this nonlocal feature [23]. Therefore, the final state ρ of the bipartite system can be expressed as

$$\rho = (1 \otimes \mathcal{S}) (M \otimes 1) |\phi\rangle\langle\phi| (M^\dagger \otimes 1). \quad (7)$$

Since the filtering operator M and the noise \mathcal{S} act only on either the first or second subsystem, it has been shown in [9] that the reduction of the entanglement of the system under the action of a noisy channel \mathcal{S} is independent of the initial state $|\chi\rangle$, and is bounded from above by the channel's action upon the maximal entangled state $|\phi\rangle$. We therefore obtain

$$C[(1 \otimes \mathcal{S})|\chi\rangle\langle\chi|] \leq \frac{d_2}{2} C[(1 \otimes \mathcal{S})|\phi\rangle\langle\phi|] C[|\chi\rangle], \quad (8)$$

where $C[\cdot]$ denotes the bipartite concurrence (5). If, moreover, the bipartite system consists of a d_1 -dimensional and a single-qubit subsystem, and just the qubit is affected by the noisy channel \mathcal{S} , the equal sign applies in inequality (8) and we obtain

$$C[(1 \otimes \mathcal{S})|\chi\rangle\langle\chi|] = C[(1 \otimes \mathcal{S})|\phi\rangle\langle\phi|] C[|\chi\rangle]. \quad (9)$$

That is, the entanglement dynamics of an arbitrary pure state of a $d_1 \otimes 2$ bipartite system is completely determined by the channel's action on the maximally entangled state

$|\phi\rangle$ of the bipartite system if the single-qubit subsystem is affected by the noisy channel \mathcal{S} .

We can utilize Eq. (9) to derive next an evolution equation of the lower bound (6) to the three-qubit concurrence, if just one qubit is affected by some noisy channel. Later, we shall generalize this evolution equation to the case of a general N -qubit system with the same assumption that just one of its qubits is subjected to a noisy channel.

C. Dynamics of the lower bound for initially pure three-qubit states

Suppose $|\chi\rangle$ is a pure state of a three-qubit system and just one qubit undergoes the action of a channel \mathcal{S} . The final state of the three-qubit system takes the form $\rho = (1 \otimes 1 \otimes \mathcal{S})|\chi\rangle\langle\chi|$, which is equivalent to the final state of a bipartite $4 \otimes 2$ system when the second subsystem is subjected to the channel \mathcal{S} . As we mentioned above, any pure state of a bipartite system can be obtained from the maximally entangled state of the bipartite system by means of a single local filtering operation M acting on the first subsystem as $|\chi\rangle = (M \otimes 1)|\phi\rangle$. In contrast, two local filters M and M' are in general required to obtain an arbitrary pure three-qubit state $|\chi\rangle$ from a maximally entangled state of three qubits $|\phi\rangle$ by $|\chi\rangle = (M \otimes M' \otimes 1)|\phi\rangle$. For three-qubit systems, however, there are two maximally entangled states which, in the computational basis, are given by [24]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad (10)$$

$$|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle). \quad (11)$$

These two entangled states cannot be obtained from each other by means of local single-qubit unitary operations [24] and give rise to two local unitary inequivalent classes of three-qubit entangled states, the so-called GHZ- and W-classes.

Although an arbitrary pure three-qubit state $|\chi\rangle$ can be generated from one of the maximally entangled states by means of local operations [24], we first need to identify the class of states either (10) and (11) to which it belongs to. For an arbitrary (pure or mixed) three-qubit entangled state, fortunately, this is possible by following the procedure due to Dür *et al.* [24] which is simple and just includes the computation of the 3-tangle as described in Ref. [16]. It leads to the distinction that every entangled three-qubit state $|\chi\rangle$, for which the 3-tangle vanishes, belong to the W-class and can thus be obtained from the W state (11) by means of local unitary operations. In contrast, any entangled three-qubit state with nonvanishing 3-tangle is part of the GHZ-class. For a given pure three-qubit state $|\chi\rangle$, it is therefore always possible to find proper local (filtering) operations M and M' so that $|\chi\rangle$ is obtained from either (10) or

(11) by $|\chi\rangle = (M \otimes M' \otimes 1)|\phi\rangle$. Moreover, we have $|\phi\rangle \equiv |\text{GHZ}\rangle$ if $|\chi\rangle$ belongs to the GHZ-class of entanglement, and $|\phi\rangle \equiv |W\rangle$ for $|\chi\rangle$ being part of the W-class.

To summarize our discussion here, the final state of the three-qubit system when one of its qubits undergoes the action of a noisy channel \mathcal{S} is given by

$$\rho = (1 \otimes 1 \otimes \mathcal{S}) \times (M \otimes M' \otimes 1) |\phi\rangle \langle \phi| (M^\dagger \otimes (M')^\dagger \otimes 1). \quad (12)$$

where $|\phi\rangle$ is one of the maximally entangled states $\{|\text{GHZ}\rangle, |W\rangle\}$. In this equation (12), the filters M , M' and the noise \mathcal{S} act on different subsystems. This allows us to apply the evolution equation for bipartite concurrence (9) to a “bi-partite” split 12|3 of the three-qubit system. We therefore obtain

$$C^{12|3}[(1 \otimes 1 \otimes \mathcal{S})|\chi\rangle \langle \chi|] = C^{12|3}[(1 \otimes 1 \otimes \mathcal{S})|\phi\rangle \langle \phi|] C^{12|3}[|\chi\rangle \langle \chi|], \quad (13)$$

while similar relations can be obtained for the “bi-partite” concurrences $C^{13|2}$ and $C^{23|1}$ of the three-qubit system. Although the Eq. (13) has similar structure to the evolution equation for bipartite systems (9), they differ by the maximally entangled state $|\phi\rangle$ in their right-hand sides: the maximally entangled state $|\phi\rangle = \sum_{i=1}^{d_2} |i\rangle \otimes |i\rangle / \sqrt{d_2}$ of the bipartite system is to be substituted in Eq. (9), while one of the maximally entangled states (10)-(11) should be used in the right hand side of Eq. (13).

Because of the symmetry of the maximally entangled states (10)-(11) with regard to the qubits permutation we have a relation

$$\begin{aligned} C^{12|3}[(1 \otimes 1 \otimes \mathcal{S})|\phi\rangle \langle \phi|] &= C^{13|2}[(1 \otimes \mathcal{S} \otimes 1)|\phi\rangle \langle \phi|] \\ &= C^{23|1}[(\mathcal{S} \otimes 1 \otimes 1)|\phi\rangle \langle \phi|], \end{aligned} \quad (14)$$

where $|\phi\rangle = \{|\text{GHZ}\rangle, |W\rangle\}$. From Eqns. (13) and (14) it follows that for an arbitrary pure three-qubit state $|\chi\rangle$ the evolution of the bipartite concurrence is independent on a bipartite cut of the three-qubit system, i.e

$$\begin{aligned} C^{12|3}[(1 \otimes 1 \otimes \mathcal{S})|\chi\rangle \langle \chi|] &= C^{13|2}[(1 \otimes \mathcal{S} \otimes 1)|\chi\rangle \langle \chi|] \\ &= C^{23|1}[(\mathcal{S} \otimes 1 \otimes 1)|\chi\rangle \langle \chi|]. \end{aligned} \quad (15)$$

Substituting the evolution equation (13) into definition of the lower bound (6) and taking into account relation (14), we finally obtain an evolution equation of the lower bound for three-qubit concurrence

$$\tau_3[(1 \otimes 1 \otimes \mathcal{S})|\chi\rangle \langle \chi|] = \tau_3[(1 \otimes 1 \otimes \mathcal{S})|\phi\rangle \langle \phi|] \tau_3[|\chi\rangle \langle \chi|], \quad (16)$$

where $\tau_3[.]$ is defined in Eq. (6). The entanglement dynamics of an arbitrary pure state $|\chi\rangle$ of a three-qubit system, when one of its qubits undergoes the action of an arbitrary noisy channel \mathcal{S} , is subjected to the dynamics of one of the maximally entangled states $|\phi\rangle =$

$\{|\text{GHZ}\rangle, |W\rangle\}$. The choice between the maximally entangled states should be done after determining the entanglement class of the given state $|\chi\rangle$ following the procedure in Ref. [24] and briefly discussed above. We note, that due to Eq. (15) the entanglement dynamics of a pure three-qubit state $|\chi\rangle$ is independent of which of the qubits is affected by the noise. In fact, this equation (15) significantly simplifies the calculation of the lower bound (6). It is sufficient to compute just one bipartite concurrence in definition (6) of the lower bound, for example $C^{12|3}[|\chi\rangle \langle \chi|]$, while the bipartite concurrences $C^{13|2}[|\chi\rangle \langle \chi|]$ and $C^{23|1}[|\chi\rangle \langle \chi|]$ are equal to it due to Eq. (15).

D. Remarks on the evolution of pure N -qubit and mixed three-qubit states

It is desirable, of course, to generalize the evolution equation (16) of the lower bound to the three-qubit concurrence also to N -qubit states, if just one of the qubits is affected by a noisy channel \mathcal{S} . In contrast to the classification of the three-qubit states, however, it is not known until now how many and which entanglement classes eventually exist for qubit systems with $N > 4$, while some classification is available for $N = 4$ [25, 26]. It is therefore not directly possible to generalize Eq. (16) for arbitrary pure states of N qubits. Nevertheless, some entanglement classes are known also for general pure N -qubit states, such as the GHZ- and W-class. If a given (pure) N -qubit state $|\chi\rangle$ belongs to the GHZ- or W-class, the evolution equation (16) of the lower bound can be extended to

$$\begin{aligned} \tau_N[(1^{\otimes N-1} \otimes \mathcal{S})|\chi\rangle \langle \chi|] &= \\ \tau_N[(1^{\otimes N-1} \otimes \mathcal{S})|\phi\rangle \langle \phi|] \tau_N[|\chi\rangle \langle \chi|], \end{aligned} \quad (17)$$

where $|\phi\rangle$ denotes the corresponding maximal entangled N -qubit state [27]

$$|\text{GHZ}\rangle_N = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \quad (18)$$

$$|W\rangle_N = \frac{1}{\sqrt{N}} (|10, \dots, 0\rangle + |01, \dots, 0\rangle + |00, \dots, 1\rangle). \quad (19)$$

We can further analyze the lower bound (6) to the three-qubit concurrence in order to understand the entanglement evolution in those cases where one starts already with an initially mixed state ρ_0 . Exploiting the convexity of the lower bound (16) as a valid entanglement measure, we have $\tau_3[(1 \otimes 1 \otimes \mathcal{S})\rho_0] = \tau_3[\sum_i p_i (1 \otimes 1 \otimes \mathcal{S})|\psi_i\rangle \langle \psi_i|] \leq \sum_i p_i \tau_3[(1 \otimes 1 \otimes \mathcal{S})|\psi_i\rangle \langle \psi_i|]$. Making use of this inequality in Eq. (16), we obtain

$$\tau_3[(1 \otimes 1 \otimes \mathcal{S})\rho_0] \leq \tau_3[(1 \otimes 1 \otimes \mathcal{S})|\phi\rangle \langle \phi|] \tau_3[\rho_0] \quad (20)$$

for the evolution of the lower bound and for an initially mixed state. As before, we assume here that just one of the qubits is affected by the noisy channel \mathcal{S} . Moreover,

the inequality (20) can be generalized for local two- and three-sided channels, i.e. to cases in which two or even all three qubits are affected by some local noise. For example, for a local two-sided channel $\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes 1 = (\mathcal{S}_1 \otimes 1 \otimes 1)(1 \otimes \mathcal{S}_2 \otimes 1)$ we find

$$\tau_3[(\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes 1)\rho_0] \leq \tau_3[(\mathcal{S}_1 \otimes 1 \otimes 1)|\phi\rangle\langle\phi|] \times \tau_3[(1 \otimes \mathcal{S}_2 \otimes 1)|\phi\rangle\langle\phi|] \tau_3[\rho_0]. \quad (21)$$

It is this particular form of Eq. (21) that gives rise to a sufficient criterion for finite-time disentanglement of arbitrary initial states being subjected to local multi-sided channels [8].

III. DYNAMICS OF THE INITIALLY MIXED STATE

The three evolution equations (16), (20) and (21) provide us with a powerful tool in order to describe the time-dependent entanglement dynamics of a three-qubit system under local noise. Although the assumption was made that only one qubit is affected by a noisy channel [apart from Eq. (21)], nothing more need to be known explicitly about the time evolution of the system's state. In the present section, we make use of Eq. (20) in analyzing the entanglement dynamics of the initially mixed state (1) for a variety of particular single-qubit noise models. Using the definition (10)-(11) of the $|\text{GHZ}\rangle$ and $|W\rangle$ in Eq. (1), we can compute the various parts of inequality (20) analytically when one qubit undergoes the action of noisy channels.

Indeed, there are several reasons for studying the entanglement evolution of the mixed state (1). For this state, first of all, an analytical expression is known for the convex roof to the concurrence (3) [14]. This enables one to compare the time-dependent lower bound from the evolution equation (20) with the behavior of the convex roof as deduced from the state dynamics under the influence of a certain noise model. Second, the mixed state density matrix (1) has simply rank two. As we have recently shown in particular examples, for rank-2 density matrices the lower bound (6) coincides with the convex roof [5]. Third, it will be quite easy to compare also the speed of ‘disentanglement’ for an (initially) pure GHZ state $\rho(t, p = 1)$ and the pure W state $\rho(t, p = 0)$ under decoherence. We also note that the values for the lower bound for these pure states are related to each other through the ratio

$$\frac{\tau_3(\rho(t=0, p=1))}{\tau_3(\rho(t=0, p=0))} = \frac{\tau_3(|\text{GHZ}\rangle)}{\tau_3(|W\rangle)} = \frac{3}{2\sqrt{2}}, \quad (22)$$

a result that was obtained in Ref. [2] by means of a lower bound to the concurrence, different from definition (6).

In the evolution equation (20) we need to obtain the final state of the three-qubit system ρ_{fin} under the influence of a local noise \mathcal{S} , i.e. $\rho_{\text{fin}} = (1 \otimes 1 \otimes \mathcal{S})\rho_{\text{ini}}$, where

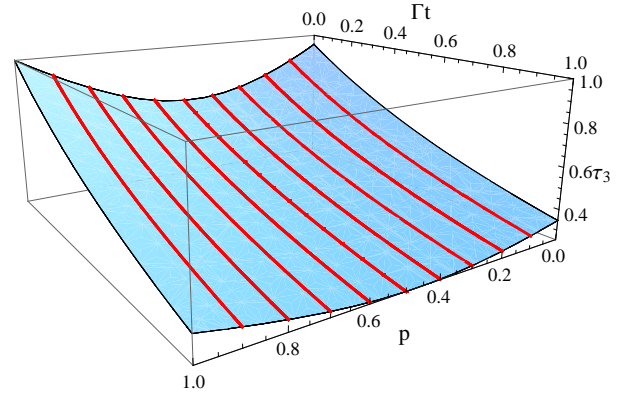


FIG. 1: (Color online) Evolution of the lower bound $\tau_3(\rho)$ for initially mixed state (1) if affected by the phase damping channel. While the blue surface shows the lhs of inequality (20), the red lines represents its rhs.

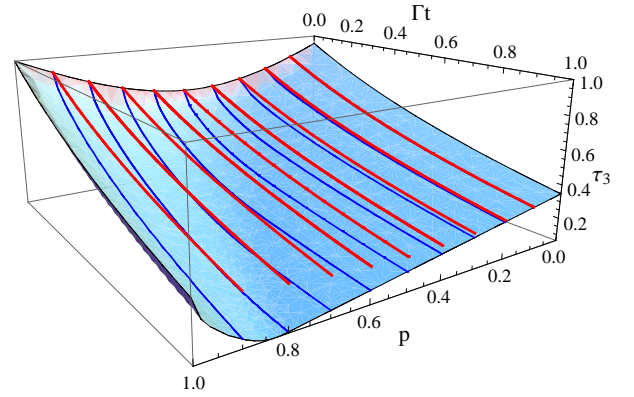


FIG. 2: (Color online) The same as in Fig. 1 but for the amplitude damping channel.

$\rho_{\text{ini}} = \{\rho_0, |\phi\rangle\langle\phi|\}$. Following quantum operation formalism [1], the final state can be obtained with the help of (Kraus) operators [28] as

$$\rho_{\text{fin}} = \sum_i K_i \rho_{\text{ini}} K_i^\dagger. \quad (23)$$

and the condition $\sum_i K_i^\dagger K_i \leq I$ is fulfilled. The Kraus operators are well known for various single-qubit noise models; in the next section, we consider two frequently used noises, namely the phase and the amplitude damping channels which are associated with relevant physical processes.

A. Phase and amplitude damping channels

Let us start the discussion with a three-qubit system which is prepared initially in the state (1) and for which just one qubits undergoes the action of the phase damping channel. A phase damping describes for instance a diffusive scattering interaction of the qubit with its en-

vironment and is known to result into a loss of phase coherence information [1]. A possible representation of the phase damping in terms of time-dependent (Kraus) operators is given by [1]

$$K_1^{\text{pd}} = \begin{pmatrix} e^{-\Gamma t} & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2^{\text{pd}} = \begin{pmatrix} \sqrt{1 - e^{-2\Gamma t}} & 0 \\ 0 & 0 \end{pmatrix}. \quad (24)$$

where Γ denotes a coupling constant. For this noise model, Fig. 1 displays the time-dependent evolution of the lower bound τ_3 for the initially prepared state (1) for different parameters p of the mixed state and at different times of the system-channel coupling. In this figure, the blue surface displays the left-hand side (lhs) of the inequality (20), while the red lines shows the corresponding right-hand side. For all parameters $0 \leq p \leq 1$ of the mixed state, the lower bound τ_3 decays exponentially and vanishes only asymptotically for $t \rightarrow \infty$. For the phase damping channel, moreover, the lhs and rhs of (20) are always equal for an arbitrary parameter p and for all times t .

If the same system is affected by a (local) amplitude damping channel, which describes the dissipative coupling of a qubit to a thermal reservoir in the zero-temperature limit [1], the operator elements in Eq. (23) are given by

$$K_1^{\text{ad}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\Gamma t} \end{pmatrix}, \quad K_2^{\text{ad}} = \begin{pmatrix} 0 & \sqrt{1 - e^{-2\Gamma t}} \\ 0 & 0 \end{pmatrix}. \quad (25)$$

For such an amplitude damping, the time evolution for the lower bound τ_3 differs from the corresponding dynamics in the phase damping channel as seen from Fig. 2. In this amplitude damping model, in particular, the (blue) surface for the lhs of (20) differs significantly from the rhs of this inequality for some values of the parameter p .

B. Sudden death of three-qubit entanglement

Based on violation of additivity in the case of entanglement decay in a multi-qubit system coupled to two independent weak noises, entanglement sudden death [4] reveals a practically important aspect of time-dependent entanglement evolution. It is important to verify whether such a phenomenon can be predicted with the suggested evolution equation of the lower bound (20). Let a three-qubit system be prepared in the state (1) and one of the qubits is subjected to the generalized amplitude damping channel. This noisy channel can be viewed as a ‘superposition’ of two independent amplitude damping channels acting on a qubit and can be expressed by four Kraus operators $K_1^{\text{gad}} = \frac{1}{2}K_1^{\text{ad}}$, $K_2^{\text{gad}} = \frac{1}{2}K_2^{\text{ad}}$ and

$$K_3^{\text{gad}} = \frac{1}{2} \begin{pmatrix} e^{-\Gamma t} & 0 \\ 0 & 1 \end{pmatrix}, \quad K_4^{\text{gad}} = \frac{1}{2} (K_2^{\text{ad}})^\dagger, \quad (26)$$

where K_1^{ad} and K_2^{ad} are defined by Eq. (25). The evolution of the lower bound τ_3 for the three-qubit state (1)

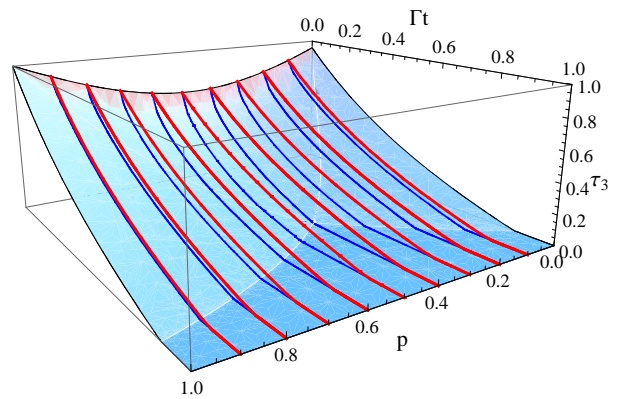


FIG. 3: (Color online) The same as in Fig. 1 but for the generalized amplitude damping channel. The sudden death of entanglement is clearly seen for all $0 \leq p \leq 1$.

is shown in Fig. 3. Although the lhs and the rhs of the inequality (20) differ for some parameters p of the initial state $\rho(p)$, they both vanish in a finite time for all values p .

IV. CONCLUSION

Unlike the evolution equations of concurrence for two qubits [8] and for a bipartite system [9], we have presented an evolution equation of the lower bound for multi-qubit concurrence (16) for an initially pure state of the system and when just one qubit is affected by local noise. This evolution equation (16) has been also extended to the cases of initially mixed states (20) and the action of many-sided noisy channels (21). In addition, the evolution equation of the lower bound for initially mixed states (20) has been employed especially to show entanglement dynamics of the mixed state (1) when just one qubit is affected by the phase, the amplitude or the generalized amplitude damping channel [cf. Figures 1–3].

Of course, the lower bound (4) is only an approximation to the convex roof for multi-qubit concurrence (3) for which an analytically computable expression is currently unavailable. Nevertheless, we have recently shown that this lower bound coincided with the convex roof for some scenarios of entanglement dynamics [5]. A more detailed analysis of the accuracy of the lower bound approximation is presently under work.

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